

ORDER OF STARLIKENESS AND CONVEXITY OF CERTAIN INTEGRAL TRANSFORMS USING DUALITY TECHNIQUES

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ABSTRACT. For $\alpha \geq 0$, $\beta < 1$ and $\gamma \geq 0$, the class $\mathcal{W}_\beta(\alpha, \gamma)$ satisfies the condition

$$\operatorname{Re} \left(e^{i\phi} ((1 - \alpha + 2\gamma)f/z + (\alpha - 2\gamma)f' + \gamma zf'' - \beta) \right) > 0, \quad \phi \in \mathbb{R}, z \in \mathbb{D};$$

is taken into consideration. The Pascu class of ξ -convex functions of order σ ($M(\sigma, \xi)$), having analytic characterization

$$\operatorname{Re} \frac{\xi z(zf'(z))' + (1 - \xi)zf'(z)}{\xi zf'(z) + (1 - \xi)f(z)} > \sigma, \quad 0 \leq \sigma < 1, \quad z \in \mathbb{D},$$

unifies starlike and convex functions class of order σ . The admissible and sufficient conditions on $\lambda(t)$ are investigated so that the integral transforms

$$V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt,$$

maps the function from $\mathcal{W}_\beta(\alpha, \gamma)$ into $M(\sigma, \xi)$. Further several interesting applications, for specific choice of $\lambda(t)$ are discussed which are related to the classical integral transform.

1. INTRODUCTION

Consider the class \mathcal{A} of all normalized and analytic function f satisfying $f(0) = f'(0) - 1 = 0$, in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let \mathcal{S} denotes the subclass of \mathcal{A} consisting of the univalent functions in \mathbb{D} . A function $f(z) \in \mathcal{S}$ is said to be starlike (\mathcal{S}^*), if $f(\mathbb{D})$ is a domain which is starlike with respect to the origin. Further generalization of the class \mathcal{S}^* is the class $\mathcal{S}^*(\sigma)$ having analytic characterization

$$\mathcal{S}^*(\sigma) = \left\{ f \in \mathcal{S} : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \sigma; \quad 0 \leq \sigma < 1, z \in \mathbb{D} \right\}.$$

Corresponding to the class $\mathcal{S}^*(\sigma)$, is the class of convex functions of order σ , ($\mathcal{C}(\sigma)$), satisfying the Alexander transform [6] that when $f \in \mathcal{C}(\sigma) \iff zf' \in \mathcal{S}^*(\sigma)$. Note that $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}(0) = \mathcal{C}$. For a function $f \in \mathcal{A}$, $M(\sigma, \xi)$ denotes the Pascu class of ξ -convex functions of order $\sigma < 1$, if it satisfies the analytic condition [11]

$$\operatorname{Re} \frac{\xi z(zf'(z))' + (1 - \xi)zf'(z)}{\xi zf'(z) + (1 - \xi)f(z)} > \sigma, \quad 0 \leq \xi \leq 1,$$

or equivalently

$$\xi zf'(z) + (1 - \xi)f(z) \in \mathcal{S}^*(\sigma)$$

Note that $M(\sigma, 0) \equiv \mathcal{S}^*(\sigma)$ and $M(\sigma, 1) \equiv \mathcal{C}(\sigma)$ which implies that the Pascu class $M(\sigma, \xi)$ unifies the class of starlike and convex functions of order σ i.e., it is the convex

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combination of the class of starlike and convex functions. It is interesting to note that the class $M(\sigma, \xi)$ also contains some non-univalent functions.

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are in \mathcal{A} then the convolution (Hadamard product) of f and g is given by

$$h(z) := (f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

For a non-negative and real-valued integrable function $\lambda(t)$, satisfying $\int_0^1 \lambda(t) dt = 1$ and a function $f \in \mathcal{A}$, R. Fournier and S. Ruscheweyh [7] introduced the integral operator

$$F(z) := V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt = z \int_0^1 \frac{\lambda(t)}{1-tz} dt * \frac{f(z)}{z}. \quad (1.1)$$

For specific choice of $\lambda(t)$, integral operator (1.1) reduces to the well-known operators such as Bernardi, Komatu, Hohlov operators and several other operators that are to be discussed in Section 4. See [5, 7, 8, 12] and references therein for these literature of these operators.

For given $\alpha \geq 0$, $\gamma \geq 0$ and $\beta < 1$, Ali et al. [1] defined the class

$$\mathcal{W}_\beta(\alpha, \gamma) = \left\{ f \in \mathcal{A} : \operatorname{Re} \left(e^{i\phi} \left((1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z) - \beta \right) \right) > 0, z \in \mathbb{D} \right\}$$

for some $\phi \in \mathbb{R}$. Many authors applied the duality theory [13, 14] on this class and its particular cases. They obtained relation between β and $\lambda(t)$ so that the integral operator given in (1.1) is univalent or belongs to $M(\sigma, \xi)$ for particular values of σ and ξ . Initially this work was motivated by R. Fournier and S. Ruscheweyh [7] by obtaining the conditions so that $V_\lambda(\mathcal{W}_\beta(1, 0)) \in M(0, 0)$. Further, the conditions under which $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(0, 0)$ was discussed by the second author with R.M. Ali et al. in [1] and the results corresponding to convexity case i.e., $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(0, 1)$ was given by R.M. Ali et al. in [2]. Generalization of the results given in [1] and [2] exhibiting the conditions under which $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(0, \xi)$ was obtained by the authors of the present work in [5]. Note that the present work is not the direct extension of the results given in [5] as the conditions obtained in Section 3 differs from the respective one given in [5]. Further details in this regard are provided in Section 4.

Recently S. Verma et al. [15] (see also [9]) investigated the constraints such that $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(\sigma, 0)$ and corresponding convexity results was given by R. Omar et al. [10] (see also [16]) under which $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(\sigma, 1)$. For further details in this direction we refer to [1, 2, 5, 12] and references therein.

The main aim of this work is to investigate the condition on $\lambda(t)$ using duality technique so that the integral transform, $V_\lambda(f) \in M(\sigma, \xi)$ whenever $f \in \mathcal{W}_\beta(\alpha, \gamma)$. This requires certain preliminaries that are outlined in Section 2. In Section 3, the necessary and sufficient conditions are derived which ensures that $V_\lambda(f)(z)$ carries the function $f(z)$ from $\mathcal{W}_\beta(\alpha, \gamma)$ into $M(\sigma, \xi)$. The criteria for $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma))$ to be in $M(\sigma, \xi)$ is simplified for further applications. In Section 4, using the sufficient condition, several interesting applications for specific choice of $\lambda(t)$ are discussed.

2. PRELIMINARIES

We consider some of the preliminaries that are useful for further discussion. Considering two constants $\mu \geq 0$ and $\nu \geq 0$ which was introduced in [1], satisfying

$$\mu + \nu = \alpha - \gamma \quad \text{and} \quad \mu\nu = \gamma. \quad (2.1)$$

For $\gamma = 0$, choose $\mu = 0$, so $\nu = \alpha \geq 0$. Now for the case $\alpha = 1 + 2\gamma$, equation (2.1) gives $\mu + \nu = 1 + \gamma = 1 + \mu\nu$, or $(\mu - 1)(1 - \nu) = 0$ which give rise to two cases:

- (i) If $\gamma > 0$, then for $\mu = 1$, gives $\nu = \gamma$.
- (ii) If $\gamma = 0$, then for $\mu = 0$, gives $\nu = \alpha = 1$.

Note: Since the case $\gamma = 0$ is considered in [12], we will only consider the case when $\gamma > 0$.

Let

$$\phi_{\mu,\nu}(z) = 1 + \sum_{n=1}^{\infty} \frac{(n\mu + 1)(n\nu + 1)}{(n + 1)} z^n, \quad (2.2)$$

and

$$\psi_{\mu,\nu}(z) = \phi_{\mu,\nu}^{-1}(z) = 1 + \sum_{n=1}^{\infty} \frac{(n + 1)}{(n\mu + 1)(n\nu + 1)} z^n = \int_0^1 \int_0^1 \frac{1}{(1 - s^\mu t^\nu z)^2} ds dt, \quad (2.3)$$

where $\phi_{\mu,\nu}^{-1}$ gives the convolution inverse of $\phi_{\mu,\nu}$ such that $\phi_{\mu,\nu} * \phi_{\mu,\nu}^{-1} = 1/(1 - z)$. For $\gamma = 0$, implies $\mu = 0$ and $\nu = \alpha$. So it become clear that

$$\psi_{0,\alpha}(z) = 1 + \sum_{n=1}^{\infty} \frac{n + 1}{n\alpha + 1} z^n = \int_0^1 \frac{1}{(1 - t^\alpha z)^2} dt.$$

The case $\gamma > 0$, implies $\mu > 0$ and $\nu > 0$. Now changing the variables $u = t^\nu$ and $v = s^\mu$ will give

$$\psi_{\mu,\nu}(z) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{u^{1/\nu-1} v^{1/\mu-1}}{(1 - uvz)^2} du dv.$$

Therefore, we can write $\psi_{\mu,\nu}$ as

$$\psi_{\mu,\nu}(z) = \begin{cases} \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{u^{1/\nu-1} v^{1/\mu-1}}{(1 - uvz)^2} du dv, & \gamma > 0, \\ \int_0^1 \frac{1}{(1 - t^\alpha z)^2} dt, & \gamma = 0, \alpha \geq 0. \end{cases}$$

Consider $g(t)$ to be the solution of initial-value problem

$$\frac{d}{dt} t^{1/\nu} (1 + g(t)) = \begin{cases} \frac{2}{\mu\nu} t^{1/\nu-1} \int_0^1 \frac{1 - \sigma(1 + st)}{(1 - \sigma)(1 + st)^2} s^{1/\mu-1} ds, & \gamma > 0, \\ \frac{2}{\alpha} t^{1/\alpha-1} \frac{1 - \sigma(1 + t)}{(1 - \sigma)(1 + t)^2}, & \gamma = 0, \alpha > 0, \end{cases} \quad (2.4)$$

satisfying $g(0) = 1$. Solution of differential equation (2.4) is given by

$$g(t) = \begin{cases} \frac{2}{\mu\nu} \int_0^1 \int_0^1 \frac{1 - \sigma(1 + swt)}{(1 - \sigma)(1 + swt)^2} s^{1/\mu-1} w^{1/\nu-1} ds dw - 1, & \gamma > 0, \\ \frac{2}{\alpha} t^{-1/\alpha} \int_0^t \frac{1 - \sigma(1 + s)}{(1 - \sigma)(1 + s)^2} s^{1/\alpha-1} ds - 1, & \gamma = 0, \alpha > 0. \end{cases} \quad (2.5)$$

Further consider $q(t)$ be the solution of the initial-value problem

$$\frac{d}{dt}(t^{1/\nu} q(t)) = \begin{cases} \frac{1}{\mu\nu} t^{1/\nu-1} \int_0^1 \frac{1 - \sigma - (1 + \sigma)st}{(1 - \sigma)(1 + st)^3} s^{1/\mu-1} ds, & \gamma > 0, \\ \frac{1}{\alpha} t^{1/\alpha-1} \frac{1 - \sigma - (1 + \sigma)t}{(1 - \sigma)(1 + t)^3}, & \gamma = 0, \alpha > 0, \end{cases} \quad (2.6)$$

satisfying $q(0) = 0$. Solving differential equation (2.6) gives

$$q(t) = \begin{cases} \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{1 - \sigma - (1 + \sigma)swt}{(1 - \sigma)(1 + swt)^3} s^{1/\mu-1} w^{1/\nu-1} ds dw, & \gamma > 0, \\ \frac{1}{\alpha} t^{-1/\alpha} \int_0^t \frac{1 - \sigma - (1 + \sigma)s}{(1 - \sigma)(1 + s)^3} s^{1/\alpha-1} ds, & \gamma = 0, \alpha > 0. \end{cases} \quad (2.7)$$

S. Verma et al. [15] and R. Omar et al. [10] (see also [9, 16]) have established the necessary and sufficient conditions under which the integral operator $V_\lambda(f(z))$ carries the function $f(z)$ from $\mathcal{W}_\beta(\alpha, \gamma)$, to the classes $\mathcal{S}^*(\sigma)$ and $\mathcal{C}(\sigma)$, respectively, which are given in the following two results.

Theorem 2.1. [9, 15] *If $\mu \geq 0$, $\nu \geq 0$ satisfies (2.1), and $\beta < 1$ is given by*

$$\frac{\beta}{1 - \beta} = - \int_0^1 \lambda(t) g(t) dt,$$

where $g(t)$ is the solution of differential equation (2.4). Assume further that $t^{1/\nu} \Lambda_\nu(t) \rightarrow 0$, and $t^{1/\mu} \Pi_{\mu, \nu}(t) \rightarrow 0$ as $t \rightarrow 0^+$. Then $F(z) := V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in \mathcal{S}^*(\sigma)$ if and only if

$$\begin{cases} \operatorname{Re} \int_0^1 \Pi_{\mu, \nu}(t) t^{1/\mu-1} \left(\frac{h_\sigma(tz)}{tz} - \frac{1 - \sigma(1 + t)}{(1 - \sigma)(1 + t)^2} \right) dt \geq 0, & \gamma > 0, \\ \operatorname{Re} \int_0^1 \Pi_{0, \alpha}(t) t^{1/\alpha-1} \left(\frac{h_\sigma(tz)}{tz} - \frac{1 - \sigma(1 + t)}{(1 - \sigma)(1 + t)^2} \right) dt \geq 0, & \gamma = 0, \end{cases} \quad (2.8)$$

where Λ_ν , $\Pi_{\mu, \nu}$ and h_σ are defined as

$$\Lambda_\nu(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\nu}} dx, \quad \nu > 0, \quad (2.9)$$

$$\Pi_{\mu, \nu}(t) = \begin{cases} \int_t^1 \Lambda_\nu(x) x^{1/\nu-1-1/\mu} dx, & \gamma > 0 \quad (\mu > 0, \nu > 0), \\ \Lambda_\alpha(t), & \gamma = 0 \quad (\mu = 0, \nu = \alpha > 0), \end{cases} \quad (2.10)$$

and

$$h_\sigma(z) = \frac{z \left(1 + \frac{\epsilon + 2\sigma - 1}{2(1 - \sigma)} z \right)}{(1 - z)^2}, \quad |\epsilon| = 1 \quad (2.11)$$

respectively.

Theorem 2.2. [10, 16] If $\mu \geq 0$, $\nu \geq 0$ satisfies (2.1), and $\beta < 1$ is given by

$$\frac{\beta - 1/2}{1 - \beta} = - \int_0^1 \lambda(t) q(t) dt,$$

where $q(t)$ is the solution of differential equation (2.6). Further $\Lambda_\nu(t)$, $\Pi_{\mu,\nu}(t)$ and h_σ are given by equation (2.9), (2.10) and (2.11). Assume that $t^{1/\mu}\Lambda_\nu(t) \rightarrow 0$, and $t^{1/\nu}\Pi_{\mu,\nu}(t) \rightarrow 0$ as $t \rightarrow 0^+$. Then

$$\begin{cases} \operatorname{Re} \int_0^1 \Pi_{\mu,\nu}(t) t^{1/\mu-1} \left(h'_\sigma(tz) - \frac{1 - \sigma - (1 + \sigma)t}{(1 - \sigma)(1 + t)^3} \right) dt \geq 0, & \gamma > 0, \\ \operatorname{Re} \int_0^1 \Pi_{0,\alpha}(t) t^{1/\alpha-1} \left(h'_\sigma(tz) - \frac{1 - \sigma - (1 + \sigma)t}{(1 - \sigma)(1 + t)^3} \right) dt \geq 0, & \gamma = 0, \end{cases} \quad (2.12)$$

if and only if $F(z) := V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in \mathcal{C}(\sigma)$.

3. MAIN RESULTS

In the following result the condition under which the integral transform $V_\lambda(f)(z)$ carries the function $f(z)$ from the class $\mathcal{W}_\beta(\alpha, \gamma)$ to $M(\sigma, \xi)$ is obtained.

Theorem 3.1. Let $\mu \geq 0$, $\nu \geq 0$, satisfies (2.1) and $\beta < 1$ is given as

$$\frac{\beta}{1 - \beta} = - \int_0^1 \lambda(t) \left((1 - \xi)g(t) + \xi(2q(t) - 1) \right) dt, \quad (3.1)$$

where $g(t)$ and $q(t)$ are defined by the differential equation (2.4) and (2.6) respectively. Further $\Lambda_\nu(t)$, $\Pi_{\mu,\nu}$ and h_σ are given by (2.9), (2.10) and (2.11). Assume that $t^{1/\nu}\Lambda_\nu(t) \rightarrow 0$ and $t^{1/\mu}\Pi_{\mu,\nu}(t) \rightarrow 0$ as $t \rightarrow 0^+$. Then $\mathcal{M}_{\Pi_{\mu,\nu}}(h_\sigma) \geq 0$ if and only if $F := V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(\sigma, \xi)$ or $(\xi z F' + (1 - \xi)F) \in \mathcal{S}^*(\sigma)$, where

$$\mathcal{M}_{\Pi_{\mu,\nu}}(h_\sigma) = \begin{cases} \int_0^1 t^{1/\mu-1} \Pi_{\mu,\nu}(t) \mathcal{L}_{\sigma,\xi,z}(t) dt, & \gamma > 0, \\ \int_0^1 t^{1/\alpha-1} \Pi_{0,\alpha}(t) \mathcal{L}_{\sigma,\xi,z}(t) dt, & \gamma = 0, \end{cases}$$

and

$$\mathcal{L}_{\sigma,\xi,z}(t) = (1 - \xi) \left(\operatorname{Re} \frac{h_\sigma(tz)}{tz} - \frac{1 - \sigma(1 + t)}{(1 - \sigma)(1 + t)^2} \right) + \xi \left(\operatorname{Re} h'_\sigma(tz) - \frac{1 - \sigma - (1 + \sigma)t}{(1 - \sigma)(1 + t)^3} \right).$$

The value of β is sharp.

Proof. The case $\gamma = 0$ was considered by the second author in [12, Theorem 2.1], so we consider here only the case $\gamma > 0$. Consider

$$H(z) := (1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z).$$

Using (2.1) in above equality gives

$$H(z) = \mu \nu z^{1-1/\mu} \left(z^{1/\mu-1/\nu+1} (z^{1/\nu-1} f(z))' \right)'. \quad (3.2)$$

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Now using (2.2) and (2.3), (3.2) is equivalent to

$$H(z) = f'(z) * \phi_{\mu,\nu}(z) \Rightarrow f'(z) = H(z) * \psi_{\mu,\nu}(z). \quad (3.3)$$

Consider $G(z) := (H(z) - \beta)/(1 - \beta)$. Therefore $\operatorname{Re}(e^{i\phi}G(z)) > 0$.

Using the duality technique given in [13] it is easy to see that $G(z) = (1 + xz)/(1 + yz)$, where $|x| = |y| = 1$. Hence

$$H(z) = \left((1 - \beta) \left(\frac{1 + xz}{1 + yz} \right) + \beta \right). \quad (3.4)$$

From (3.3) and (3.4) gives

$$f'(z) = \left((1 - \beta) \left(\frac{1 + xz}{1 + yz} \right) + \beta \right) * \psi_{\mu, \nu}(z).$$

Integrating the above expression gives

$$\frac{f(z)}{z} = \frac{1}{z} \int_0^z \left((1 - \beta) \left(\frac{1 + xw}{1 + yw} \right) + \beta \right) dw * \psi_{\mu, \nu}(z).$$

For $F \in M(\sigma, \xi) \iff (\xi z F' + (1 - \xi)F) \in \mathcal{S}^*(\sigma)$. So it is sufficient to check the condition of starlikeness. By the well known result from convolution theory [13, Pg 94],

$$(\xi z F' + (1 - \xi)F) \in \mathcal{S}^*(\sigma) \iff \frac{1}{z} \left(((1 - \xi)F + \xi z F') * h_{\sigma}(z) \right) \neq 0$$

where

$$h_{\sigma}(z) = \frac{z \left(1 + \frac{\epsilon + 2\sigma - 1}{2(1 - \sigma)} z \right)}{(1 - z)^2}, \quad |\epsilon| = 1, \quad |z| < 1.$$

Now $F \in M(\sigma, \xi)$ if and only if

$$\begin{aligned} 0 &\neq \frac{1}{z} \left((1 - \xi) \left(\int_0^1 \lambda(t) \frac{f(tz)}{t} dt * \frac{h_{\sigma}(z)}{z} \right) + \xi \left(\int_0^1 \lambda(t) \frac{f(tz)}{t} dt * z h'_{\sigma}(z) \right) \right) \\ &= (1 - \xi) \left(\int_0^1 \frac{\lambda(t)}{1 - tz} dt * \frac{f(z)}{z} * \frac{h_{\sigma}(z)}{z} \right) + \xi \left(\int_0^1 \frac{\lambda(t)}{1 - tz} dt * \frac{f(z)}{z} * h'_{\sigma}(z) \right) \\ &= (1 - \xi) \left(\int_0^1 \frac{\lambda(t)}{1 - tz} dt * \left(\frac{1}{z} \int_0^z (1 - \beta) \left(\frac{1 + x\omega}{1 + y\omega} \right) d\omega + \beta \right) * \psi(z) * \frac{h_{\sigma}(z)}{z} \right) \\ &\quad + \xi \left(\int_0^1 \frac{\lambda(t)}{1 - tz} dt * \left(\frac{1}{z} \int_0^z (1 - \beta) \left(\frac{1 + x\omega}{1 + y\omega} \right) d\omega + \beta \right) * \psi(z) * h'_{\sigma}(z) \right) \\ &= (1 - \xi) \left(\int_0^1 \lambda(t) \frac{h_{\sigma}(tz)}{tz} dt * (1 - \beta) \left(\frac{1}{z} \int_0^z \frac{1 + x\omega}{1 + y\omega} d\omega + \frac{\beta}{1 - \beta} \right) * \psi(z) \right) \\ &\quad + \xi \left(\int_0^1 \lambda(t) h'_{\sigma}(tz) dt * (1 - \beta) \left(\frac{1}{z} \int_0^z \frac{1 + x\omega}{1 + y\omega} d\omega + \frac{\beta}{1 - \beta} \right) * \psi(z) \right) \\ &= (1 - \beta) \left[(1 - \xi) \int_0^1 \lambda(t) \left(\frac{1}{z} \int_0^z \frac{h_{\sigma}(t\omega)}{t\omega} d\omega + \frac{\beta}{1 - \beta} \right) dt \right. \\ &\quad \left. + \xi \int_0^1 \lambda(t) \left(\frac{1}{z} \int_0^z h'_{\sigma}(t\omega) d\omega + \frac{\beta}{1 - \beta} \right) dt \right] * \psi(z) * \frac{1 + xz}{1 + yz}, \end{aligned}$$

which clearly holds if and only if [13, Pg 23]

$$\begin{aligned} \operatorname{Re} \left((1-\beta) \left[(1-\xi) \int_0^1 \lambda(t) \left(\frac{1}{z} \int_0^z \frac{h_\sigma(t\omega)}{t\omega} d\omega + \frac{\beta}{1-\beta} \right) dt \right. \right. \\ \left. \left. + \xi \int_0^1 \lambda(t) \left(\frac{1}{z} \int_0^z h'_\sigma(t\omega) d\omega + \frac{\beta}{1-\beta} \right) dt \right] * \psi(z) \right) > \frac{1}{2}. \end{aligned}$$

Using (2.8) and (2.12) from Theorem 2.1 and Theorem 2.2, the above inequality is equivalent to

$$\begin{aligned} \int_0^1 t^{1/\mu-1} \Pi_{\mu,\nu}(t) \left[(1-\xi) \left(\operatorname{Re} \frac{h_\sigma(tz)}{tz} - \frac{1-\sigma(1+t)}{(1-\sigma)(1+t)^2} \right) \right. \\ \left. + \xi \left(\operatorname{Re} h'_\sigma(tz) - \frac{1-\sigma-(1+\sigma)t}{(1-\sigma)(1+t)^3} \right) \right] dt \geq 0, \end{aligned}$$

which directly implies that $F \in M(\sigma, \xi) \iff \mathcal{M}_{\Pi_{\mu,\nu}}(h_\sigma) \geq 0$.

Now to prove the sharpness, let $f(z) \in \mathcal{W}_\beta(\alpha, \gamma)$ be the solution of the differential equation

$$(1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma zf''(z) = \beta + (1-\beta)\frac{1+z}{1-z} \quad (3.5)$$

where β satisfies (3.1). Using series expansion of (2.5) and (2.7) in (3.1) gives

$$\frac{\beta}{1-\beta} = -1 - \frac{2}{(1-\sigma)} \left[(1-\xi) \sum_{n=1}^{\infty} \frac{(n+1-\sigma)(-1)^n \tau_n}{(1+\mu n)(1+\nu n)} + \xi \sum_{n=1}^{\infty} \frac{(n+1)(n+1-\sigma)(-1)^n \tau_n}{(1+\mu n)(1+\nu n)} \right] \quad (3.6)$$

$$\text{where } \tau_n = \int_0^1 \lambda(t) t^n dt.$$

If

$$F := V_\lambda(f(z)) \in M(\sigma, \xi) \iff K(z) := (\xi z F' + (1-\xi)F) \in \mathcal{S}^*(\sigma). \quad (3.7)$$

Using (2.1) and (3.5) gives

$$f(z) = z + \sum_{n=1}^{\infty} \frac{2(1-\beta)}{(n\mu+1)(n\nu+1)} z^{n+1},$$

which further gives

$$F := V_\lambda(f(z)) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt = z + \sum_{n=1}^{\infty} \frac{2(1-\beta)\tau_n}{(n\mu+1)(n\nu+1)} z^{n+1}.$$

The above equation implies that

$$K(z) := (1-\xi) \left(z + \sum_{n=1}^{\infty} \frac{2(1-\beta)\tau_n}{(n\mu+1)(n\nu+1)} z^{n+1} \right) + \xi \left(z + \sum_{n=1}^{\infty} \frac{2(1-\beta)(n+1)\tau_n}{(n\mu+1)(n\nu+1)} z^{n+1} \right). \quad (3.8)$$

Using (3.6) and (3.8) gives

$$\begin{aligned} zK'(z)|_{z=-1} &= -1 - (1-\xi) \sum_{n=1}^{\infty} \frac{2(1-\beta)(n+1)\tau_n(-1)^n}{(n\mu+1)(n\nu+1)} \\ &\quad - \xi \sum_{n=1}^{\infty} \frac{2(1-\beta)(n+1)^2\tau_n(-1)^n}{(n\mu+1)(n\nu+1)} \\ &= \sigma K(-1). \end{aligned}$$

Therefore $(zK'(z))/(K(z)) = \sigma$ at $z = -1$, which clearly indicates that the result is sharp. \square

Particular values of Theorem 3.1 reduce to several known results.

Remark 3.1. (1) For $\xi = 0$, Theorem 3.1 reduces to [15, Theorem 3.1] (see also [9, Theorem 2.1]) and for $\xi = 1$, Theorem 3.1 gives [16, Theorem 3.1] (see also [10, Theorem 2.2]).

(2) When $\sigma = 0$, Theorem 3.1 reduces to [5, Theorem 2.1].

The necessary and sufficient conditions so that the integral operator given in (1.1) carries the function from $\mathcal{W}_\beta(\alpha, \gamma)$ into the class $M(\sigma, \xi)$ is obtained in Theorem 3.1, is not an easy one to use for the applications. Hence for the application purpose an easier sufficient condition is presented in the following theorem.

Theorem 3.2. Let $\sigma \in [0, 1/2]$ and assume that Λ_ν and $\Pi_{\mu, \nu}$ are integrable on $[0, 1]$ and positive on $(0, 1)$. If $\beta < 1$ satisfy (3.1) and

$$\frac{\xi t^{1/\xi-1/\mu+1} d(t^{1/\mu-1/\xi} \Pi_{\mu, \nu}(t))}{(\log(1/t))^{1+2\sigma}} \quad (3.9)$$

is increasing on $(0, 1)$, for $\mu \geq 1$, $0 \leq \xi \leq 1$. Then $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(\sigma, \xi)$.

Proof. Consider

$$\begin{aligned} \mathcal{M}_{\Pi_{\mu, \nu}}(h_\sigma) &= \int_0^1 t^{1/\mu-1} \Pi_{\mu, \nu}(t) \left[(1-\xi) \left(\operatorname{Re} \frac{h_\sigma(tz)}{tz} - \frac{1-\sigma(1+t)}{(1-\sigma)(1+t)^2} \right) \right. \\ &\quad \left. + \xi \left(\operatorname{Re} h'_\sigma(tz) - \frac{(1-\sigma-(1-\sigma)t)}{(1-\sigma)(1+t)^3} \right) \right] dt \\ &= \int_0^1 t^{1/\mu-1} \Pi_{\mu, \nu}(t) \left[(1-\xi) \left(\operatorname{Re} \frac{h_\sigma(tz)}{tz} - \frac{1-\sigma(1+t)}{(1-\sigma)(1+t)^2} \right) \right. \\ &\quad \left. + \xi \frac{d}{dt} \left(\operatorname{Re} \frac{h_\sigma(tz)}{tz} - \frac{t(1-\sigma(1+t))}{(1-\sigma)(1+t)^2} \right) \right] dt. \end{aligned}$$

A simple computation gives

$$\mathcal{M}_{\Pi_{\mu, \nu}}(h_\sigma) = \int_0^1 t^{1/\mu-1} \left(\left(1 - \frac{\xi}{\mu} \right) \Pi_{\mu, \nu}(t) - \xi t \Pi'_{\mu, \nu}(t) \right) \left(\operatorname{Re} \frac{h_\sigma(tz)}{tz} - \frac{1-\sigma(1+t)}{(1-\sigma)(1+t)^2} \right) dt.$$

The function $t^{1/\mu-1}$ decreases on $(0, 1)$, when $\mu \geq 1$. Therefore the condition (3.9) along with [3, Theorem 1.3] gives $\mathcal{M}_{\Pi_{\mu, \nu}}(h_\sigma) \geq 0$. So the desired result follows from Theorem 3.1. \square

To ensure the sufficiency of Theorem 3.2 for the integral transform to be in $M(\sigma, \xi)$ by an easier method, the following results are obtained.

Since the case $\gamma = 0$ was considered in [12], we only consider the case $\gamma > 0$. In [12], for the case $\gamma = 0$ the condition $\lambda(1) = 0$ was assumed. To prove that the Theorem 3.1 holds true for the case $\gamma > 0$, we need to show that the condition

$$p(t) := \frac{\left(1 - \frac{\xi}{\mu} \right) \Pi_{\mu, \nu}(t) + \xi t^{1/\nu-1/\mu} \Lambda_\nu(t)}{(\log(1/t))^{1+2\sigma}}$$

is decreasing on $(0, 1)$, where $\Lambda_\nu(t)$ and $\Pi_{\mu, \nu}(t)$ are defined in (2.9) and (2.10). For this it is enough to prove that $p'(t) \leq 0$.

Now $p(t) = k(t)/(\log(1/t))^{1+2\sigma}$ where $k(t) = \left(1 - \frac{\xi}{\mu}\right) \Pi_{\mu,\nu}(t) + \xi t^{1/\nu-1/\mu} \Lambda_\nu(t)$. The condition $p'(t) \leq 0$ is equivalent of obtaining

$$\frac{p'(t)}{p(t)} = \frac{k'(t)}{k(t)} + \frac{(1+2\sigma)}{t(\log(1/t))} \leq 0,$$

since $p(t) \geq 0$ for $t \in (0, 1)$. In order to show that $p'(t) \leq 0$, is similar to obtain

$$\begin{aligned} r(t) &:= k(t) + \frac{t \log(1/t) k'(t)}{(1+2\sigma)} \\ &= \left(1 - \frac{\xi}{\mu}\right) \Pi_{\mu,\nu}(t) + \xi t^{1/\nu-1/\mu} \Lambda_\nu(t) \\ &\quad - \frac{\log(1/t) \left(\left(1 - \frac{\xi}{\nu}\right) t^{1/\nu-1/\mu} \Lambda_\nu(t) + \xi t^{1-1/\mu} \lambda(t) \right)}{(1+2\sigma)} \leq 0, \quad t \in (0, 1). \end{aligned}$$

As $r(1) = 0$, in order to prove that $p'(t) \leq 0$, it is enough to show that $r(t)$ is an increasing function on $(0, 1)$. We compute $r'(t)$ explicitly and after an easy computation $r'(t) \geq 0$ is equivalent to the inequality

$$\begin{aligned} s(t) &:= \left(1 - \frac{\xi}{\nu}\right) \left(\left(\frac{1}{\mu} - \frac{1}{\nu}\right) \log(1/t) - 2\sigma \right) t^{1/\nu-1-1/\mu} \Lambda_\nu(t) \\ &\quad + \left(\left(1 - \xi + \frac{\xi}{\mu} - \frac{\xi}{\nu}\right) \log(1/t) - 2\sigma \xi \right) t^{-1/\mu} \lambda(t) - \xi \log(1/t) t^{1-1/\mu} \lambda'(t) \geq 0. \end{aligned} \quad (3.10)$$

At $t = 1$, $s(t) \leq 0$ because ξ and σ are positive terms. So we assume that $\lambda(t) = 0$. Hence $s(t) = 0$ at $t = 1$. The function $s(t)$ is positive if $s(t)$ is a decreasing function of $t \in (0, 1)$ i.e., $s'(t) \leq 0$.

Using (3.10), $s'(t)$ is equivalent to

$$s'(t) = L(t) t^{1/\nu-2-1/\mu} \Lambda_\nu(t) + M(t) t^{-1-1/\mu} \lambda(t) + N(t) t^{-1/\mu} \lambda'(t) + P(t) t^{1-1/\mu} \lambda''(t) \leq 0$$

where

$$\begin{aligned} L(t) &:= \left(1 - \frac{\xi}{\nu}\right) \left[\left(1 + \frac{1}{\mu} - \frac{1}{\nu}\right) \left(2\sigma - \left(\frac{1}{\mu} - \frac{1}{\nu}\right) \log(1/t)\right) - \left(\frac{1}{\mu} - \frac{1}{\nu}\right) \right] \\ M(t) &:= - \left[\left(1 - \xi\right) \frac{1}{\mu} + \left(\frac{1}{\mu} - \frac{1}{\nu}\right) \left(1 + \frac{\xi}{\mu} - \frac{\xi}{\nu}\right) \right] \log(1/t) + (1 - 2\sigma) \left(1 + \frac{\xi}{\mu} - \frac{\xi}{\nu}\right) - \xi \\ N(t) &:= \left(\left(1 - 2\xi + \frac{2\xi}{\mu} - \frac{\xi}{\nu}\right) \log(1/t) + \xi(1 - 2\sigma) \right) \end{aligned}$$

and $P(t) := -\xi \log(1/t)$.

From (2.1), $\mu \geq 1$ implies $\nu \geq \mu \geq 1$. For $\xi \in [0, 1]$, the term $(1 - \xi/\nu)$ and $t^{1/\nu-1-1/\mu} \Lambda_\nu(t)$ are non-negative for $t \in (0, 1)$. The function $s'(t) \leq 0$, if $L(t)$, $M(t)$, $N(t)$ and $P(t)$ are negative. For

$$\sigma \leq \frac{1}{2} \left(\frac{1/\mu - 1/\nu}{1 + 1/\mu - 1/\nu} \right), \quad \mu \geq 1,$$

implies $L(t) \leq 0$ and $M(t) \leq 0$.

Now to show that $s'(t) \leq 0$ for $t \in (0, 1)$, it is enough to prove that

$$\begin{aligned} \xi t \log(1/t) \lambda''(t) - \left(\left(1 - 2\xi + \frac{2\xi}{\mu} - \frac{\xi}{\nu} \right) \log(1/t) + \xi(1 - 2\sigma) \right) \lambda'(t) &\geq 0 \\ \iff \frac{t\lambda''(t)}{\lambda'(t)} &\geq \left(\frac{1}{\xi} - 2 + \frac{2}{\mu} - \frac{1}{\nu} \right) + \frac{(1 - 2\sigma)}{\log(1/t)}, \end{aligned} \quad (3.11)$$

for $\mu \geq 1$ and $\sigma \leq \frac{1}{2} \left(\frac{1/\mu - 1/\nu}{1 + 1/\mu - 1/\nu} \right)$.

Now we are in a position to state the general result.

Theorem 3.3. *Let $\beta < 1$ is defined by (3.1). Then $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(\sigma, \xi)$, if*

$$\frac{t\lambda''(t)}{\lambda'(t)} \geq \left(\frac{1}{\xi} - 2 + \frac{2}{\mu} - \frac{1}{\nu} \right) + \frac{(1 - 2\sigma)}{\log(1/t)}, \quad (3.12)$$

for $\mu \geq 1$, $\gamma > 0$ and $\sigma \leq \frac{1}{2} \left(\frac{1/\mu - 1/\nu}{1 + 1/\mu - 1/\nu} \right)$.

4. APPLICATIONS

In this section, the number of applications for well-known integral operators are considered and the conditions are obtained.

Remark 4.1. *Note that $\lambda(1) = 0$ is assumed in Theorem 3.2. For the case $\lambda(t) = (c + 1)t^{c-1}$, it is not possible to get $\lambda(1) = 0$ and hence the result corresponding to Bernardi integral transform cannot be obtained using Theorem 3.2. But the admissibility $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(0, \xi)$ for the Bernardi integral operator was possible in [5, Theorem 12], as this condition was not required in the admissibility to the class of $M(0, \xi)$ in [5].*

Consider

$$\lambda(t) = Dt^{B-1}(1-t)^{C-A-B}\omega(1-t), \quad (4.1)$$

where the function $\omega(1-t) = 1 + \sum_{n=1}^{\infty} x_n(1-t)^n$, for $t \in (0, 1)$ and $x_n \geq 0$. D is chosen

such that it satisfies normalization condition $\int_0^1 \lambda(t)dt = 1$. An easy computation on $\lambda(t)$ defined by (4.1), obtain $\lambda'(t)$ and $\lambda''(t)$. Substituting in (3.11) will give

$$Dt^{B-2}(1-t)^{C-A-B-2}[X(t)\omega(1-t) + t(1-t)Y(t)\omega'(1-t) + t^2(1-t)^2\omega''(1-t)] \geq 0,$$

for $t \in (0, 1)$, where

$$\begin{aligned} X(t) := &\log(1/t)((B-1)(B-2) + 2(1-B)(C-A-2)t + (C-A-B)(C-A+B-3)t^2) \\ &+ ((C-A-B)t + (1-B)(1-t)) \left((1-2\sigma) + \left(\frac{1}{\xi} - 2 + \frac{2}{\mu} - \frac{1}{\nu} \right) \log(1/t) \right) \end{aligned}$$

$$Y(t) := 2\log(1/t)((C-A-B)t + (1-B)(1-t)) + \left((1-2\sigma) + \left(\frac{1}{\xi} - 2 + \frac{2}{\mu} - \frac{1}{\nu} \right) \log(1/t) \right).$$

The inequality (3.12) will hold true, if $X(t)$ and $Y(t)$ are non-negative terms for some values of A , B and C . When $B \leq 1$, $C \geq A + 3$ and $\left(\frac{1}{\xi} + \frac{2}{\mu} - \frac{1}{\nu} \right) \geq 2$, then Theorem 3.3 is true.

The above observation results in the following theorem.

Theorem 4.1. For $\gamma > 0$, $\mu \geq 1$, $\xi \in [0, 1]$ and $\beta < 1$ is defined by (3.1), where $\lambda(t) = Dt^{B-1}(1-t)^{C-A-B}\omega(1-t)$. Then $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(\sigma, \xi)$, if $B \leq 1$, $C \geq A+3$ and $((1/\xi) + (2/\mu) - (1/\nu)) \geq 2$, for

$$0 \leq \sigma \leq \frac{1}{2} \left(\frac{1/\mu - 1/\nu}{1 + 1/\mu - 1/\nu} \right).$$

Theorem 4.2. Let $a, b, c > 0$, $\gamma > 0$ and $\beta < 1$ satisfy

$$\frac{\beta}{(1-\beta)} = -R \int_0^1 t^{b-1}(1-t)^{c-a-b} {}_2F_1 \left(\begin{matrix} c-a, & 1-a \\ c-a-b+1 \end{matrix}; 1-t \right) [(1-\xi)g(t) + \xi(2q(t)-1)]dt, \quad (4.2)$$

where $R = \Gamma(c)/(\Gamma(a)\Gamma(b)\Gamma(c-a-b+1))$. Then $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(\sigma, \xi)$ if $c \geq a+3$, $b \leq 1$, $\left(\frac{1}{\xi} + \frac{2}{\mu} - \frac{1}{\nu}\right) \geq 2$ and $0 \leq \sigma \leq \frac{1}{2} \left(\frac{1/\mu - 1/\nu}{1 + 1/\mu - 1/\nu} \right)$. The value of β is sharp.

Proof. Choosing

$$\omega(1-t) = R {}_2F_1 \left(\begin{matrix} c-a, & 1-a \\ c-a-b+1 \end{matrix}; 1-t \right),$$

and substitute a, b, c instead of A, B, C respectively in Theorem 4.1 will give the required result.

In order to obtain sharpness take the extremal function $f(z)$ of the class $\mathcal{W}_\beta(\alpha, \gamma)$ as

$$f(z) = z + 2 \sum_{n=2}^{\infty} \frac{(1-\beta)}{(1+(n-1)\mu)(1+(n-1)\nu)} z^n.$$

Consider

$$\Omega_n := \int_0^1 t^{n+b-2}(1-t)^{c-a-b} {}_2F_1 \left(\begin{matrix} c-a, & 1-a \\ c-a-b+1 \end{matrix}; 1-t \right) dt.$$

Using (1.1) and

$$\lambda(t) := R t^{b-1}(1-t)^{c-a-b} {}_2F_1 \left(\begin{matrix} c-a, & 1-a \\ c-a-b+1 \end{matrix}; 1-t \right),$$

it follows that

$$F(z) := z + 2(1-\beta)R \sum_{n=2}^{\infty} \frac{\Omega_n}{(1+(n-1)\mu)(1+(n-1)\nu)} z^n,$$

which implies that

$$zF'(z) = z + 2(1-\beta)R \sum_{n=2}^{\infty} \frac{n\Omega_n}{(1+(n-1)\mu)(1+(n-1)\nu)} z^n.$$

Using (3.7) gives

$$K(z) = z + 2(1-\beta)R \sum_{n=2}^{\infty} \frac{((n-1)\xi + 1)\Omega_n}{(1+(n-1)\mu)(1+(n-1)\nu)} z^n. \quad (4.3)$$

For the case $\gamma > 0$, the series representation of $g(t)$ defined in (2.5) is given by

$$g(t) = 2 \sum_{n=0}^{\infty} \frac{(n+1-\sigma)(-t)^n}{(1-\sigma)(1+n\mu)(1+n\nu)} - 1, \quad (4.4)$$

and $q(t)$ defined in (2.7) is given by

$$q(t) = \sum_{n=0}^{\infty} \frac{(n+1)(n+1-\sigma)(-t)^n}{(1-\sigma)(1+n\mu)(1+n\nu)}. \quad (4.5)$$

From (4.4) and (4.5)
 $(1-\xi)g(t) + \xi(2q(t) - 1)$

$$= 1 + 2 \sum_{n=2}^{\infty} \frac{(1+\xi(n-1))(n-\sigma)(-t)^{n-1}}{(1-\sigma)(1+(n-1)\mu)(1+(n-1)\nu)} \quad (4.6)$$

$$= 2 {}_5F_4 \left(1, \frac{1}{\mu}, \frac{1}{\nu}, (2-\sigma), \left(1 + \frac{1}{\xi}\right) ; \left(1 + \frac{1}{\mu}\right), \left(1 + \frac{1}{\nu}\right), (1-\sigma), \frac{1}{\xi} ; -t \right) - 1. \quad (4.7)$$

Using (4.2) and (4.6) gives

$$\frac{1}{2(1-\beta)} = R \sum_{n=2}^{\infty} \frac{(1+\xi(n-1))(n-\sigma)(-1)^{n+1}}{(1-\sigma)(1+(n-1)\mu)(1+(n-1)\nu)} \Omega_n. \quad (4.8)$$

From (4.3) and (4.8), as $z \rightarrow -1$ gives

$$\left. \frac{zK'(z)}{K(z)} \right|_{z=-1} = \sigma,$$

which means that the result is sharp. \square

Remark 4.2.

- (1) When $\sigma = 0$, Theorem 4.2 gives different conditions and a precise bound on b than in [5, Theorem 3.4].
- (2) A particular instance, when $\xi = 0$, Theorem 4.2 gives a result with the smaller range for σ than the result given in [15, Theorem 5.1] (see also [9, Theorem 3.3]).

The result for the case, $\gamma = 0$, $\sigma = 0$ and $\xi = 0$ is obtained in [4, Theorem 1]. In Theorem 4.2, the conditions are obtained when $\gamma > 0$, hence the results cannot be compared.

Corollary 4.1. Let $a, b, c > 0$, $\gamma > 0$ and $\beta_0 < \beta < 1$, where

$$\beta_0 = 1 - \frac{1}{2 \left(1 - {}_6F_5 \left(\begin{matrix} 1, b, \frac{1}{\mu}, \frac{1}{\nu}, (2-\sigma), \left(1 + \frac{1}{\xi}\right) \\ c, \left(1 + \frac{1}{\mu}\right), \left(1 + \frac{1}{\nu}\right), (1-\sigma), \frac{1}{\xi} \end{matrix} ; -1 \right) \right)}.$$

Then $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(\sigma, \xi)$ if $c \geq 4$, $b \leq 1$, $\left(\frac{1}{\xi} + \frac{2}{\mu} - \frac{1}{\nu}\right) \geq 2$ and $0 \leq \sigma \leq \frac{1}{2} \left(\frac{1/\mu - 1/\nu}{1 + 1/\mu - 1/\nu} \right)$.

Proof. Consider $a = 1$ in (4.2). Using (4.7) gives

$$\frac{\beta}{1-\beta} = -\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \left({}_2F_4 \left(\begin{matrix} 1, \frac{1}{\mu}, \frac{1}{\nu}, (2-\sigma), \left(1+\frac{1}{\xi}\right) \\ \left(1+\frac{1}{\mu}\right), \left(1+\frac{1}{\nu}\right), (1-\sigma), \frac{1}{\xi} \end{matrix} ; -t \right) - 1 \right) dt,$$

which is equivalent to

$$\begin{aligned} \frac{\beta - 1/2}{1-\beta} &= -\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} {}_5F_4 \left(\begin{matrix} 1, \frac{1}{\mu}, \frac{1}{\nu}, (2-\sigma), \left(1+\frac{1}{\xi}\right) \\ \left(1+\frac{1}{\mu}\right), \left(1+\frac{1}{\nu}\right), (1-\sigma), \frac{1}{\xi} \end{matrix} ; -t \right) dt \\ &= -\sum_{n=0}^{\infty} \frac{(1)_n (b)_n \left(\frac{1}{\mu}\right)_n \left(\frac{1}{\nu}\right)_n (2-\sigma)_n \left(1+\frac{1}{\xi}\right)_n}{(c)_n \left(1+\frac{1}{\mu}\right)_n \left(1+\frac{1}{\nu}\right)_n (1-\sigma)_n \left(\frac{1}{\xi}\right)_n (1)_n} (-1)^n \\ &= -{}_6F_5 \left(\begin{matrix} 1, b, \frac{1}{\mu}, \frac{1}{\nu}, (2-\sigma), \left(1+\frac{1}{\xi}\right) \\ c, \left(1+\frac{1}{\mu}\right), \left(1+\frac{1}{\nu}\right), (1-\sigma), \frac{1}{\xi} \end{matrix} ; -1 \right). \end{aligned}$$

By the given hypothesis and applying Theorem 3.1 will give the required result. \square

Theorem 4.3. Let $-1 < c \leq 0$, $\delta \geq (3-c)$, $\gamma > 0$ ($\mu \geq 1$) and $\beta < 1$ satisfy

$$\frac{\beta}{(1-\beta)} = -\frac{(1+c)^\delta}{\Gamma(\delta)} \int_0^1 t^c (\log(1/t))^{\delta-1} [(1-\xi)g(t) + \xi(2q(t)-1)] dt.$$

Then $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(\sigma, \xi)$, for

$$\left(\frac{1}{\xi} + \frac{2}{\mu} - \frac{1}{\nu} \right) \geq 2 \quad \text{and} \quad 0 \leq \sigma \leq \frac{1}{2} \left(\frac{1/\mu - 1/\nu}{1 + 1/\mu - 1/\nu} \right).$$

Proof. Consider $B = c + 1$, $C - A - B = \delta - 1$ and

$$\omega(1-t) = \left(\frac{\log(1/t)}{(1-t)} \right)^{\delta-1}.$$

Then the required conclusion follows from Theorem 4.2. \square

Remark 4.3.

- (1) For a particular value of $\sigma = 0$ [5, Theorem 3.6] give weak result for δ in comparison to the result obtained from Theorem 4.3.
- (2) When $\xi = 0$, [15, Theorem 5.4] (see also [9, Theorem 3.2]) give weak bounds for the parameters than the result obtained from Theorem 4.3.
- (3) When $\xi = 1$, Theorem 4.3, improves the result obtained in [16, Theorem 5.9].

Theorem 4.4. Let $\mu \geq 1$, $\gamma > 0$, $0 < \xi \leq 1$ and $\beta < 1$ satisfy (3.1), where

$$\lambda(t) = \begin{cases} (a+1)(b+1) \frac{t^a(1-t^{b-a})}{b-a}, & b \neq a, \\ (a+1)^2 t^a \log(1/t), & b = a. \end{cases} \quad (4.9)$$

Then $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(\sigma, \xi)$, provided $-1 < a \leq 0$ for $a = b$ or $a \neq b$ and

$$0 \leq \sigma \leq \frac{1}{2} \left(\frac{1/\mu - 1/\nu}{1 + 1/\mu - 1/\nu} \right).$$

Proof. Using $\lambda(t)$ given in (4.9) gives

$$\frac{t\lambda''(t)}{\lambda'(t)} = \begin{cases} \frac{(a(a-1) - b(b-1)t^{b-a})}{(a - bt^{b-a})}, & b \neq a, \\ \frac{(1 - 2a + a(a-1)\log(1/t))}{(-1 + a\log(1/t))}, & b = a. \end{cases}$$

Case(i): Consider $a = b > -1$ and $\gamma > 0$. Substituting the values of $t\lambda''(t)/\lambda'(t)$ in (3.12) and on further simplification gives

$$U(t)(\log(1/t))^2 + V(t)(\log(1/t)) + (1 - 2\sigma) \geq 0, \quad (4.10)$$

where

$$U(t) := a \left(a + 1 - \frac{1}{\xi} - \frac{2}{\mu} + \frac{1}{\nu} \right),$$

$$V(t) := \left(\frac{1}{\xi} + \frac{2}{\mu} - \frac{1}{\nu} - 1 - a(3 - 2\sigma) \right).$$

As $(1 - 2\sigma) > 0$, then (4.10) holds true if $U(t)$ and $V(t)$ are non-negative on the given conditions. Clearly the inequality (4.10) under the hypotheses is true.

Case(ii): Consider $-1 < a < b$ and $\gamma > 0$. Substituting the value of $t\lambda''(t)/\lambda'(t)$ in (3.12) is equivalent to $\phi_t(a) \geq \phi_t(b)$, $t \in (0, 1)$, where

$$\phi_t(a) := a(a-1)t^a \log(1/t) - a \left(\left(\frac{1}{\xi} + \frac{2}{\mu} - \frac{1}{\nu} - 2 \right) \log(1/t) + (1 - 2\sigma) \right) t^a.$$

We will claim that $\phi_t(a)$ is a decreasing function of a for $a \in (-1, 0]$. Differentiating $\phi_t(a)$ with respect to a gives

$$\phi'_t(a) := -t^a (R + S \log(1/t) + T (\log(1/t))^2)$$

where

$$R := (1 - 2\sigma),$$

$$S := \left(\frac{1}{\xi} + \frac{2}{\mu} - \frac{1}{\nu} - 1 + a(2\sigma - 3) \right),$$

$$T := a \left(a - \frac{1}{\xi} - \frac{2}{\mu} + \frac{1}{\nu} + 1 \right).$$

The function $\phi'_t(a) \leq 0$, if the terms R , S and T are non-negative, which is clearly true by the hypothesis when $a \in (0, 1]$. Hence the desired conclusion follows. \square

Remark 4.4.

- (1) In Theorem 4.4, consider the case when $\xi = 0$. We obtain the bound for a as, $-1 < a \leq 0$; $a = b$ or $a < b$. This range for a is smaller than the range obtain in [15, Theorem 5.3] (see also [9, Theorem 3.2]). For the case when $\xi = 1$, Theorem 4.4 provides weaker bounds for a when compared to [16, Theorem 5.6].

- (2) Consider $\xi = 1$ and $\sigma = 0$. The result obtained by Theorem 4.4 has same bound for the case $a = b \leq 0$ and the result for $a < b$ coincides in [2, Theorem 5.7].

Theorem 4.5. Let $\mu \geq 1$, $\gamma > 0$, $0 < \xi \leq 1$ and $\beta < 1$ satisfy (3.1), where

$$\lambda(t) = \frac{(1-k)(3-k)}{2} t^{-k}(1-t^2), \quad 0 \leq k < 1. \quad (4.11)$$

Then $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(\sigma, \xi)$, if $k = (1 - 1/\xi - 2/\mu + 1/\nu)$ and $\sigma = 1/2$.

Proof. Using $\lambda(t)$ given in (4.11), we have

$$\frac{t\lambda''(t)}{\lambda(t)} = -\frac{[k(1+k) - (2-k)(1-k)t^2]}{[k + (2-k)t^2]}.$$

Therefore to obtain the result, it suffice to prove that

$$-\frac{[k(1+k) - (2-k)(1-k)t^2]}{[k + (2-k)t^2]} \geq \left(\frac{1}{\xi} - 2 + \frac{2}{\mu} - \frac{1}{\nu}\right) + \frac{(1-2\sigma)}{\log(1/t)},$$

under the given hypothesis. Using the fact that $\log(1/t) \geq 2(1-t)/(1+t)$, for $t \in (0, 1)$ the result can be easily obtained. □

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